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# The pole dynamics of rational solutions of the viscous Burgers equation

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**Abstract**
Rational solutions of the viscous Burgers equation are examined using the dynamics of their poles in the complex $x$-plane. The dynamical system for the motion of these poles is finite dimensional and not Hamiltonian. Nevertheless, we show that this finite-dimensional system is completely integrable, by explicit construction of a sufficient number of conserved quantities. The dynamical system has a class of non-equilibrium similarity solutions for which all poles have equal real part for $t$ sufficiently large. Within the context of the finite-dimensional dynamical system these solutions are shown to be asymptotically stable.

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## 1. Introduction

The Burgers equation [1]

$$u_t + uu_x = \nu u_{xx} \tag{1}$$

arises in the study of gas dynamics. It is one of the canonical examples of a nonlinear partial differential equation. It is often used to regularize the dissipationless Burgers equation $u_t + uu_x = 0$, to avoid the formation of shocks. In the context of integrable systems it is often mentioned as the simplest nonlinear partial differential equation which is completely integrable, because of its linearizing transformation

$$u = -2\nu \frac{\phi_x}{\phi}, \tag{2}$$

  

due to Forsyth, Hopf and Cole [3, 5, 6]. This transformation transforms (1) into the linear heat equation

$$\phi_t = \nu \phi_{xx}. \tag{3}$$

The Burgers equation (1) stands out among the integrable equations as it is dissipative. Apart from its linearizing transformation it has almost no features in common with the other canonical integrable equations. One of the few was already pointed out by Chudnovs'ki and Chudnovs'ki, who treated it parallel to the Korteweg-de Vries equation

$$u_t + uu_x = u_{xxx}, \tag{4}$$

when they discussed the pole dynamics of rational, trigonometric or hyperbolic solutions of these equations in [2]. We reexamine the pole dynamics of rational solutions of (1) in more detail, demonstrating that the resulting finite-dimensional dynamical system for the motion of the poles is completely integrable, as a sufficient number of constants of the motion is constructed explicitly. As should be expected, the dynamical system governing the motion of the poles is not Hamiltonian, and a full set of constants of the motion is required.

After establishing the integrability, a class of similarity solutions of the dynamical system is found, corresponding to non-equilibrium solutions of (1). We prove that these solutions are asymptotically stable, both linearly and nonlinearly, in the context of the finite-dimensional system describing the motion of the poles.

## 2. The motion of the poles of rational solutions

By definition, a rational solution only has a finite number of poles in the complex $x$-plane. Thus we use the ansatz

$$u(x,t) = -2\nu \sum_{k=1}^{N} \frac{R_k(t)}{x - x_k(t)}, \tag{5}$$

where $N$ is the number of poles. The ansatz (5) is substituted in (1). Next, let $x = x_k + \epsilon$, and expand the resulting equation in powers of $\epsilon$. The most singular terms have $1/\epsilon^3$ behaviour. Equating their coefficient to zero gives

$$R_k(t) \equiv 1, \qquad k = 1, \ldots, N. \tag{6}$$

Thus all poles have residue 1. At order $1/\epsilon^2$, the dynamical system for the motion of the poles is obtained:

$$\dot{x_k} = -2\nu \sum_{n \neq k}^{N} \frac{1}{x_k - x_n}, \qquad k = 1, \ldots, N, \tag{7}$$

where the dot denotes differentiation with respect to $t$. The coefficients of the terms of order $1/\epsilon$ and order 1 vanish identically, thus equations (6) and (7) are the only requirements for the ansatz (5) to give a solution of Burgers equation (1). In particular, the situation is different from that of for instance the Korteweg-de Vries (KdV) equation where the poles are constrained to live on a lower dimensional solution manifold which is invariant under the dynamics. In the case of the rational solutions of the KdV equation this also imposes a constraint on the number of poles $N$ of such a solution.

**Remarks.**

- Examining the dynamics (7) with $N = 2$ (but $x_1$ not necessarily equal to $x_2^*$) shows that the poles exert a force field on each other that is attractive in the real direction and repulsive in the imaginary direction. Thus the long-term dynamics of (7) tends to line up the poles on a vertical line in the complex plane along which the poles tend to infinity.
- The calculations presented here are consistent with those of [2]. There are significant differences with the work of Senouf [8, 9], where meromorphic (in $x$) solutions of the Burgers equation (1) are considered which have an infinite number of poles. Also, the solutions in [8, 9] have an essential singularity at $t = 0$.
- Different constraints may be imposed on the solution (5) of (1): we may require those solutions to be real valued, and we may require the real-valued solutions to be bounded. On the other hand, one can study solutions of (1) that are complex valued for real $x$, or solutions that are real valued for real $x$, but unbounded on the real line. These are valid mathematical problems, although they are less relevant for the applications of the Burgers equation. The calculations we present are valid without any constraints, unless specifically stated otherwise.

  If we require solutions to be real when restricted to the real line the set $\{x_k(t), k = 1, \ldots, N\}$ is invariant under complex conjugation. Further, if we are only interested in solutions that are bounded on the real $x$-axis, none of the poles $x_k(t)$ are to be real. In that case $N$ is necessarily even, in order for solutions to be bounded.
- The dynamical system (7) presented here is similar to that describing the interaction of a finite number of point vortices [11]. That system strongly resembles (7) if one were to consider an imaginary viscosity and add some complex conjugations. The point vortex system is known to be Hamiltonian, unlike the system considered here, see below. As a consequence, the nature of the dynamics is entirely different.

## 3. A complete set of constants of the motion

The system (7) is not Hamiltonian. One easily checks that the divergence of the flow is not zero. If the systems were Hamiltonian with $N$ degrees of freedom (i.e., is $2N$ dimensional), Liouville's theorem [10] to ensure integrability demands the existence of $N$ constants of the motion that are mutually in involution with respect to the Poisson bracket. This is not sufficient here. In this section, $N$ constants of the motion are constructed for the $N$-dimensional system (7). One should note that these counting arguments require some care as the systems under consideration are defined in terms of $N$ complex-valued functions. However, the constants of the motion constructed are also complex valued.

Define

$$J_n(t) = \frac{1}{n}\sum_{k=1}^{N} x_k^n(t), \qquad n = 1, \ldots, N. \tag{8}$$

From the fundamental theorem of algebra, it is clear that the quantities $J_1, J_2, \ldots, J_N$ are functionally independent, as they are a basis for the symmetric functions of $x_1, x_2, \ldots, x_N$ of degree $\leqslant N$. We have the following theorem:

**Theorem.** *The quantities $J_1, J_2, \ldots, J_N$ are polynomial in $t$. This $t$ dependence is completely determined by the recursion relationship*

$$\frac{\mathrm{d}J_n}{\mathrm{d}t} = -\nu(n-2)(2N+1-n)J_{n-2} - \nu\sum_{k=1}^{n-3} k(n-2-k)J_k J_{n-2-k}, \tag{9}$$

*and* $J_1 = J_{10}$, $J_2 = -\nu N(N-1)t + J_{20}$. *For* $n = 3$ *the sum in the recursion relationship is to be ignored. Here* $J_{10}, J_{20}, \ldots, J_{N0}$ *are constants of the motion, determined by the initial position of the poles* $x_1, x_2, \ldots, x_N$, *such that* $J_n(0) = J_{n0}, n = 1, 2, \ldots, N$. *For example, we have*

$$\begin{aligned}
J_1 &= J_{10},\\
J_2 &= J_{20} - \nu N(N-1)t,\\
J_3 &= J_{30} - 2\nu(N-1)J_{10}t,\\
J_4 &= J_{40} - \nu\left(2(2N-3)J_{20} + J_{10}^2\right)t + \nu^2 N(N-1)(2N-3)t^2,\\
&\ldots
\end{aligned}$$

**Proof.**

$n = 1$. Consider

$$\begin{aligned}
\frac{\mathrm{d}J_1}{\mathrm{d}t} &= \frac{d}{\mathrm{d}t}\sum_{k=1}^{N} x_k = -2\nu\sum_{k=1}^{N}\sum_{j\neq k}^{N}\frac{1}{x_k - x_j}, \quad \text{using (7)}\\
&= -2\nu\sum_{j=1}^{N}\sum_{k\neq j}^{N}\frac{1}{x_k - x_j}, \quad \text{(switching the sums)}\\
&= 2\nu\sum_{j=1}^{N}\sum_{k\neq j}^{N}\frac{1}{x_j - x_k} = 2\nu\sum_{k=1}^{N}\sum_{j\neq k}^{N}\frac{1}{x_k - x_j}, \quad \text{(switching the labels } k \text{ and } j)\\
&= -\frac{\mathrm{d}J_1}{\mathrm{d}t},
\end{aligned}$$

and thus $\mathrm{d}J_1/\mathrm{d}t = 0$ and $J_1 = J_{10}$.

$n = 2$. Consider

$$\begin{aligned}
\frac{\mathrm{d}J_2}{\mathrm{d}t} &= \frac{d}{\mathrm{d}t}\left(\frac{1}{2}\sum_{k=1}^{N} x_k^2\right) = \sum_{k=1}^{N} x_k\dot{x}_k = -2\nu\sum_{k=1}^{N} x_k\sum_{j\neq k}^{N}\frac{1}{x_k - x_j}, \qquad \text{using (7)}\\
&= -2\nu\sum_{k=1}^{N}\sum_{j\neq k}^{N}\left(1 + \frac{x_j}{x_k - x_j}\right)\\
&= -2\nu N(N-1) - 2\nu\sum_{j=1}^{N}\sum_{k\neq j}^{N}\frac{x_k}{x_j - x_k}, \qquad \text{(switching the labels } k \text{ and } j)\\
&= -2\nu N(N-1) - \frac{\mathrm{d}J_2}{\mathrm{d}t},
\end{aligned}$$

and thus $\mathrm{d}J_2/\mathrm{d}t = -\nu N(N-1)$ and $J_2 = -\nu N(N-1)t + J_{20}$.

$n \geqslant 3$. Consider

$$\begin{aligned}
\frac{\mathrm{d}J_n}{\mathrm{d}t} &= \frac{d}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{N} x_k^n\right) = \sum_{k=1}^{N} x_k^{n-1}\dot{x}_k = -2\nu\sum_{k=1}^{N}\sum_{j\neq k}^{N}\frac{x_k^{n-1}}{x_k - x_j}, \qquad \text{using (7)}\\
&= -2\nu\sum_{k=1}^{N}\sum_{j\neq k}^{N}\left(\frac{x_k^{n-1} - x_j^{n-1}}{x_k - x_j} + \frac{x_j^{n-1}}{x_k - x_j}\right)
\end{aligned}$$

$$= -2\nu \sum_{k=1}^{N} \sum_{j\neq k}^{N} \frac{x_k^{n-1} - x_j^{n-1}}{x_k - x_j} - 2\nu \sum_{j=1}^{N} \sum_{k\neq j}^{N} \frac{x_k^{n-1}}{x_j - x_k}, \quad \text{(relabelling the second term)}$$

$$= -2\nu \sum_{k=1}^{N} \sum_{j\neq k}^{N} \frac{x_k^{n-1} - x_j^{n-1}}{x_k - x_j} - \frac{\mathrm{d}J_n}{\mathrm{d}t},$$

and thus

$$\frac{\mathrm{d}J_n}{\mathrm{d}t} = -\nu \sum_{k=1}^{N} \sum_{j\neq k}^{N} \frac{x_k^{n-1} - x_j^{n-1}}{x_k - x_j} = -\nu \sum_{k=1}^{N} \sum_{j\neq k}^{N} \sum_{m=0}^{n-2} x_k^m x_j^{n-m-2}$$

$$= -\nu \sum_{m=0}^{n-2} \sum_{k=1}^{N} x_k^m \sum_{j\neq k}^{N} x_j^{n-m-2} = -\nu \sum_{m=0}^{n-2} \sum_{k=1}^{N} x_k^m \left( \sum_{j=1}^{N} x_j^{n-m-2} - x_k^{n-m-2} \right)$$

$$= -\nu \sum_{m=0}^{n-3} \sum_{k=1}^{N} x_k^m \left( \sum_{j=1}^{N} x_j^{n-m-2} - x_k^{n-m-2} \right) - \nu(N-1) \sum_{k=1}^{N} x_k^{n-2}$$

$$= -\nu \sum_{m=0}^{n-3} \sum_{k=1}^{N} x_k^m (n-2-m) J_{n-2-m} + \nu \sum_{m=0}^{n-3} \sum_{k=1}^{N} x_k^{n-2} - \nu(N-1)(n-2) J_{n-2}$$

$$= -\nu \sum_{m=1}^{n-3} \sum_{k=1}^{N} x_k^m (n-2-m) J_{n-2-m} - \nu \sum_{k=1}^{N} (n-2) J_{n-2} + \nu (n-2)^2 J_{n-2}$$
$$- \nu(N-1)(n-2) J_{n-2}$$

$$= -\nu \sum_{m=1}^{n-3} m(n-2-m) J_m J_{n-2-m} - \nu N(n-2) J_{n-2} + \nu(n-2)^2 J_{n-2}$$
$$- \nu(N-1)(n-2) J_{n-2}$$

$$= -\nu \sum_{m=1}^{n-3} m(n-2-m) J_m J_{n-2-m} - \nu(n-2)(2N+1-n) J_{n-2},$$

which proves the theorem. □

**Remark.** A related set of conserved quantities may be derived using the Forsyth–Cole–Hopf transformation (2). The heat equation (3) may be rewritten as a formal conservation law:

$$\frac{\partial}{\partial t}\phi(x,t) = \frac{\partial}{\partial x}\nu\phi_x(x,t). \tag{10}$$

For solutions of the form (5) $\phi = \prod_{n=1}^{N}(x - x_n(t)) = \sum_{k=0}^{N} \Delta_k x^k$, where

$$\Delta_N = 1$$

$$\Delta_{N-1} = -\sum_{n=1}^{N} x_n$$

$$\Delta_{N-2} = \sum_{n=1}^{N} \sum_{m\neq n} x_n x_m$$

$$\dots$$

$$\Delta_0 = (-1)^N \prod_{n=1}^{N} x_n.$$

Note that the quantities $\Delta_k$, $k = 0, \ldots, N-1$ are symmetric functions of the pole locations $x_1, \ldots, x_N$. Their collection is a different but equivalent basis for the symmetric functions of the $N$ variables $x_1, \ldots, x_N$.

Substituting $\phi = \sum_{k=0}^{N} x^k \Delta_k$ into the conservation form of the heat equation (10) and equating coefficients of different powers of $x$ gives rise to the equations

$$\dot{\Delta}_k = \nu(k+2)(k+1)\Delta_{k+2}, \tag{11}$$

for $k = 0, \ldots, N-2$, with $\dot{\Delta}_{N-1} = 0$ and $\dot{\Delta}_N = 0$. The initial values of the quantities $\Delta_k$ give rise to a set of constants of the motion for (7) which is different from that constructed before but clearly equivalent to it. Using the Forsyth–Cole–Hopf transformation thus allows for a quicker proof of the integrability of the system (7). However, the first proof given requires no knowledge of the connection between the Burgers equation and the heat equation. Such a direct proof may prove easier to generalize in situations where such connections are not known.

## 4. Similarity solutions

A family of similarity solutions of (7) is easily constructed using the ansatz

$$x_n(t) = x_0 + T(t)\zeta_n, \qquad n = 1, \ldots, N, \tag{12}$$

with $\zeta_k, x_0 \in \mathbb{C}$. Substitution in (7) gives

$$TT' = -\frac{2\nu}{\zeta_k}\sum_{n\neq k}^{N}\frac{1}{\zeta_k - \zeta_n}, \qquad k = 1, \ldots, N. \tag{13}$$

Since the right-hand side does not depend on $t$, and the left-hand side is independent of the index $k$, both sides are equal to a constant $c$. It follows from (13) that the value of $c$ does not impact the product of $T(t)\zeta_k$. Without loss of generality we let $c = \nu$ in what follows. This gives rise to

$$T = \sqrt{2\nu(t - t_0)}, \tag{14$a$}$$

$$\zeta_k = -2\sum_{n\neq k}^{N}\frac{1}{\zeta_k - \zeta_n}, \qquad k = 1, \ldots, N. \tag{14$b$}$$

Thus, once the nonlinear algebraic system (14*b*) is solved for the set $\{\zeta_1, \ldots, \zeta_N\}$, the set of expressions (12) provides a solution of the dynamical system (7). At this point (5) may be used to reconstruct a solution of the Burgers equation (1). From (14*a*), all poles collide at $x = x_0$ when $t = t_0$, and no poles collide for $t \neq t_0$. Due to the nature of the force field, this implies that all poles are lined up horizontally for $t < t_0$. For $t > t_0$, all poles are on a vertical line through $x_0$. For both cases the poles are symmetrically distributed around $x_0$. If $x_0$ is real, the solution of the Burgers equation is real. Otherwise it is not.

**Examples.** For real-valued solutions of (1), $x_0$ merely shifts the solution (5) along the real axis, and it is omitted in the following examples.

- $N = 2$, *real-valued solutions*. In this case $\zeta_2 = \zeta_1^*$ and it easily follows that the only solution of (14*b*) (up to permutation) is $\zeta_1 = i = -\zeta_2$. Then $u = -4\nu x/(x^2 + 2\nu(t - t_0))$

is a two-pole real-valued rational solution of (1). This solution is nonsingular for real $x$ for $t > t_0$.

- $N = 4$, *real-valued solutions.* Looking for solutions with $\zeta_1 = \mathrm{i}\alpha = \zeta_2^*$, $\zeta_3 = \mathrm{i}\beta = \zeta_4^*$, with $\alpha, \beta \in \mathbb{R}^+$ and $\alpha < \beta$ gives $\alpha = \sqrt{3-\sqrt{6}}$, $\beta = \sqrt{3+\sqrt{6}}$, so that

$$x_1 = \mathrm{i}\sqrt{2\nu(t-t_0)(3-\sqrt{6})} = x_2^*, \qquad x_3 = \mathrm{i}\sqrt{2\nu(t-t_0)(3+\sqrt{6})} = x_4^*,$$

resulting in a four-pole real-valued rational solution of (1):

$$u = -\frac{4\nu x}{x^2 + 2\nu(t-t_0)(3-\sqrt{6})} - \frac{4\nu x}{x^2 + 2\nu(t-t_0)(3+\sqrt{6})}.$$

This solution is nonsingular for real $x$ for $t > t_0$.

## 5. Stability of the similarity solutions

In this section, we examine the stability of the similarity solutions specified by (12), in the context of the finite-dimensional system (7). In order to do so, we transform this system to one where the similarity solutions correspond to stationary points. Let

$$x_j = x_0 + x\sqrt{2(t-t_0)}z_j(t), \qquad j = 1, \ldots, N. \tag{15}$$

Then $z_j(t) = \zeta_j$ $(j = 1, \ldots, N)$ gives the similarity solutions. An elementary calculation gives

$$\frac{\mathrm{d}z_j}{\mathrm{d}\tau} = -\nu \sum_{k\neq j}^{N} \frac{1}{z_j - z_k} - \frac{1}{2}z_j, \qquad j = 1, \ldots, N. \tag{16}$$

Here $\tau = \ln(t - t_0)$. Since we are interested in the behaviour of this system for $t \to \infty$, $\tau$ may be considered real. Next, we linearize this nonlinear system around its stationary points, the similarity solutions. Let

$$z_j = \zeta_j + \epsilon\xi_j + \mathcal{O}(\epsilon^2), \qquad j = 1, \ldots, N. \tag{17}$$

Substitution in (16) and collecting first-order terms in $\epsilon$ results in the linear system of equations

$$\dot{\xi}_j = \left(-\frac{1}{2} + \nu\sum_{k\neq j}^{N}\frac{1}{(\zeta_j-\zeta_k)^2}\right)\xi_j - \nu\sum_{k\neq j}^{N}\frac{\xi_k}{(\zeta_j-\zeta_k)^2}, \qquad j = 1, \ldots, N. \tag{18}$$

Since the poles of the similarity solutions line up vertically as $t \to \infty$, we set

$$\zeta_j = x_0 + \mathrm{i}\beta_j, \qquad \beta_j \in \mathbb{R}, \qquad j = 1, \ldots, N,$$

resulting in

$$\dot{\xi}_j = \left(-\frac{1}{2} - \nu\sum_{i\neq j}^{N}\frac{1}{(\beta_j-\beta_i)^2}\right)\xi_j + \nu\sum_{k\neq j}^{N}\frac{\xi_k}{(\beta_j-\beta_k)^2}, \qquad j = 1, \ldots, N. \tag{19}$$

In order to examine the linear stability of the similarity solutions, we investigate the location of the eigenvalues of the matrix $\Lambda$ specified by the right-hand side of (19):

$$\Lambda_{jk} = \begin{cases} -\dfrac{1}{2} - \nu \displaystyle\sum_{i \neq j}^{N} \dfrac{1}{(\beta_j - \beta_i)^2} & \text{if} \quad k = j, \\ \nu \dfrac{1}{(\beta_j - \beta_k)^2} & \text{if} \quad k \neq j, \end{cases} \tag{20}$$

for $j, k = 1, \ldots, N$.

Let

$$r_j = \nu \sum_{k \neq j}^{N} \frac{1}{(\beta_i - \beta_j)^2} > 0, \qquad j = 1, \ldots, N. \tag{21}$$

It is an immediate consequence of Gershgorin's theorem [7] that all eigenvalues $\lambda_j$ ($j = 1, \ldots, N$) of $\Lambda$ are contained in the closed discs in $\mathbb{C}$ specified by

$$\left|\lambda - \left(-\tfrac{1}{2} - r_j\right)\right| \leqslant r_j, \qquad j = 1, \ldots, N. \tag{22}$$

Thus all eigenvalues are in the left-half plane. In fact, they are all to the left of or on the vertical line through $-1/2$. Further, since $\Lambda$ is symmetric, all its eigenvalues are real. Thus all eigenvalues are contained in the overlapping intervals

$$\left[-\tfrac{1}{2} - 2r_j, -\tfrac{1}{2}\right], \qquad j = 1, \ldots, N.$$

On the other hand, it is a simple observation to see that $\lambda = -1/2$ is an eigenvalue, with eigenvector $(1, 1, \ldots, 1)^T$. Moreover, a simple shift of the eigenvalue by any positive number strictly greater than $1/2 + \max(r_1, \ldots, r_N)$ allows us to use Perron's theorem [4], from which we conclude that this maximal eigenvalue is simple. Thus

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \mathrm{e}^{-\tau/2} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (t - t_0)^{-1/2} \tag{23}$$

is one of the fundamental solutions of (18). Further, it follows from the symmetry of $\Lambda$ that it has a complete set of eigenvectors, so that all other fundamental solutions of (18) decay faster than $\exp(-\tau/2)$, corresponding to decay in the original variable $t$.

In terms of the original variable $t$ the fundamental mode corresponding to the maximal eigenvalue is a neutral mode, reflecting the arbitrariness in choosing $x_0$. Since we know from section 3 that the centre of mass of the pole locations is conserved, this neutral mode can be eliminated by restricting the class of perturbations so that the centre of mass is constant. If the class of perturbations is not so restricted (in other words, if the pole locations of the perturbation are not symmetric with respect to the original centre of mass), then the overall solution (similarity solution plus perturbation) has a different centre of mass $\hat{x}_0$, and asymptotically tends to the original similarity solution, except it is translated by $x_0 - \hat{x}_0$.

Thus we have established the linear asymptotic stability (up to the existence of a neutral mode) of the similarity solutions through the context of the system (18). Their nonlinear asymptotic stability follows from the Hartman-Grobman theorem [10], as applied to (16).

In summary, we have proven the following theorem:

**Theorem.** *The similarity solutions specified by (12) are asymptotically stable solutions of the finite-dimensional system (7) with respect to the class of perturbations that preserve the centre of mass. If the perturbations are not restricted this way, solutions near similarity solutions still limit to potentially translated incarnations of them.*

## 6. Conclusions

In this paper, we study the rational solutions of the Burgers equation. We have established the following results:

- The dynamical system determining the motion of the poles is completely integrable. This was shown using a direct calculation, although it may also be shown using the heat equation and its connection to the Burgers equation using the Forsyth–Cole–Hopf transformation. Since the constants of the motion are related to the symmetric functions of $N$ variables in a straightforward way, we have reduced the problem of integrating the dynamical system to that of finding the roots of a degree $N$ polynomial.
- A class of similarity solutions has been constructed. For such real-valued solutions, all poles are aligned on the real axis for $t < t_0$, whereas they are aligned vertically in the complex plane for $t > t_0$. At $t = t_0$ all poles coalesce. Here $t_0$ is an arbitrary real constant. The poles approach the point of coalescence horizontally at a rate of $\sqrt{t - t_0}$, and they separate from this point vertically at the same rate.
- After transformation to a system for which the similarity solutions are stationary, they are shown to be asymptotically stable. This is done using a linear stability analysis, after which the Hartman-Grobman theorem is invoked to conclude nonlinear asymptotic stability.

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## References

[1] Burgers J M 1948 A mathematical model illustrating the theory of turbulence *Adv. Appl. Mech.* **1** 171–99
[2] Chudnovs'ki D V and Chudnovs'ki G V 1977 Pole expansions of nonlinear partial differential equations *Nuevo Cimento* B **40** 339–53
[3] Cole J 1951 On a quasilinear parabolic equation occurring in aerodynamics *Q. J. Appl. Math.* **9** 225–36
[4] Gantmacher F R 1959 *The Theory of Matrices* vol 2 (New York: Chelsea Publishing)
[5] Hopf E 1950 The partial differential equation $u_t + uu_x = \mu u_{xx}$ *Commun. Pure Appl. Math.* **3** 201–30
[6] Forsyth A R 1906 *Theory of Differential Equations* (Cambridge: Cambridge University Press)
[7] Lax P D 1997 *Linear Algebra* (New York: Wiley)
[8] Senouf D 1997 Dynamics and condensation of complex singularities for Burgers' equation I *SIAM J. Math. Anal.* **28** 1457–89
[9] Senouf D 1997 Dynamics and condensation of complex singularities for Burgers' equation II *SIAM J. Math. Anal.* **28** 1490–513
[10] Wiggins S 2003 *Introduction to Applied Nonlinear Dynamical Systems and Chaos* 2nd edn (New York: Springer)
[11] Kimura Y 1990 Parametric motion of complex-time singularity toward real collapse *Physica* D **46** 439–48